UNIT 4 PARTITIONS AND DISTRIBUTIONS

Structure **Page No.**

4.0 INTRODUCTION

In the last two units we have exposed you to a variety of combinatorial techniques. In this unit we look at a few more ways of counting arrangements of objects when order matters, and when it doesn't.

In Sec. 4.2, we focus on the ways in which a natural number can be written as a sum of natural numbers. In the process you will be introduced to a useful 'recurrence relation'.

We link this, in Sec. 4.3, with the different ways in which n objects can be distributed among m containers. As you will see, there are four broad possible kinds of distributions. In each case, we consider ways of counting all the distributions. In the process you will also be introduced to Stirling numbers.

With this unit we come to the end of our discussion on counting techniques. Some of the problems you have studied here will be looked at from different approaches in our later course MCS-033.

You should attempt the assignment based on the course after studying this unit, and this block.

4.1 OBJECTIVES

After going through this unit, you should be able to:

- define an integer partition, and count the number of partitions of an integer;
- count the number of ways of distributing distinguishable and indistinguishable objects, respectively, into distinguishable containers;
- count the number of ways of distributing distinguishable and indistinguishable objects, respectively, into indistinguishable containers.

4.2 INTEGER PARTITIONS

Suppose a detergent manufacturer wants to promote her product by giving a gift token with 100 bars out of the whole stock. The lucky persons among her customers will get the gift. Some of them may buy more than one bar at a time with the hope of getting gifts. In how many ways can the 100 gift tokens get distributed? One possible way is that all the 100 bars with gifts are bought by 100 different customers. We can indicate this situation by $100 = \underbrace{1+1+\ldots+1}_{100\text{ times}}$. Another possibility is that somebody buys 2 bars, somebody else buys 3 bars, and the remaining 95 bars are distributed amongst 95 different people. We are not interested in the order in which the bars are bought. For example, here we are not interested in whether the person who bought 2 bars bought them before the person who bought the 3 bars. So, we can indicate this situation by $100 = \underbrace{1+1+...+1}_{95\text{ times}} + 2 + 3$. More generally, we can indicate each way of distributing the 100 bars with gifts by $100 = p_1 + p_2 + p_2 + \ldots + p_k$, where the p_i are natural numbers, and $p_1 \le p_2 \le \ldots \le p_k$. Each way of writing 100 in this form is called an **integer partition** of 100. More generally, we have the following definition.

Definition: Any representation of $n \in \mathbf{N}$ as a sum of positive integers in non-increasing order is called a **partition** (or **integer partition**) of n. Each such partition can be written in the form $n = p_1 + p_2 + \ldots + p_k$, where $p_1 \le p_2 \le \ldots \le p_k$.

Here, $p_1, p_2, \ldots, p_k$ are called the **parts** of the partition, and the **number of parts** of the partition is k.

While we chose 100 in the example above, it is really a huge number in the context of integer partitions. Let us consider a smaller number, say 5. How many partitions of 5 can you think of? There are 7 altogether, namely, 5, 1+4, 2+3, 1+1+3, 1+2+2, 1+1+1+2 and 1+1+1+1+1.

In books, you will often come across the notation p(n) for the number of partitions of n.

If **we represent the number of partitions of the integer n by P_n,** we have shown that $P_5 = 7$. These partitions have 1, 2, 2, 3, 3, 4 and 5 parts, respectively.

If **we represent the number of partitions of n with exactly k parts by** P_n^k, then we have $P_5^1 = 1, P_5^2 = 2, P_5^3 = 2, P_5^4 = 1, P_5^5 = 1$.

To check your understanding of the material so far, try the following exercises.

E1) Write down all the partitions of 7. Also find P_7^4 and P_7^5.

E2) Let us consider the situation of the detergent manufacturer again. Suppose she only wants to distribute 10 gift tokens in 5 specific sales districts, where the sales are low. What is the number of ways of doing this?

You may wonder if you've found all the partitions in E2. One way to check is by finding out the required number in terms of partitions of smaller numbers, which may be easier to find. One such relation between partitions of n and n+1, n+2, etc. is given in the following theorem.

Theorem 1: $P_n^1 + P_n^2 + ... + P_n^k = P_{n+k}^k, P_n^1 = P_n^n = 1$, for $1 \le k \le n$, that is, the number of **partitions of n with at most k parts** is the same as the number of **partitions of n+k with exactly k parts**.

Before we begin the proof of this theorem, let us consider an example. Let us take n = 4, k = 3. According to Theorem 1, we must have $P_4^1 + P_4^2 + P_4^3 = P_7^3$. Note that $P_4^1 + P_4^2 + P_4^3$ is the total number of partitions of 4 with 1, 2 or 3 parts, i.e., the number of partitions with **at most** 3 parts. There is one partition of 4 with one part, 4 = 4. Let us write this as a 3-tuple, (4, 0, 0), adding two more zeroes since we are considering partitions with at most **3** parts. If we add 1 to all the entries of this 3-tuple, we get (4+1,0+1,0+1) = (5,1,1) and (1+1+5 is a partition of 7 with three parts. Similarly, consider the partition 4 = 1+3 of 4 into two parts. Again, we can write this as (1, 3, 0).

Now, if we add 1 to each of the entries, we get (2, 4, 1) and 1+2+4 is a partition of 7 into three parts. Conversely, if we take the partition 7 = 1 + 3 +3 of 7 into three parts, write it as (1, 3, 3) and **subtract** 1 from all the entries, we get (0, 2, 2) which corresponds to the partition 4 = 2+2 of 4 into 2 parts. In this way, we can match every partition of 4 with **at most 3** parts with a partition of 7 with **exactly 3** parts, and vice versa. This is the basic idea behind our proof of Theorem 1, which we now give.

Proof of Theorem 1: The cases $P_n^1 = 1 = P_n^n$ follow from the definition.

We will prove the general formula now. Let M be the set of partitions of n having k or less parts. We can consider each partition belonging to M as a k-tuple after adding as many zeroes as necessary. Define the mapping

$$(p_1, p_2,\ldots, p_m, \underbrace{0,0,\ldots,0}_{\substack{(k-m)\\ \text{times}}}) \mapsto (p_1+1,p_2+1,\ldots,p_m + 1, \underbrace{1,1,\ldots,1}_{\substack{(k-m)\\ \text{times}}}), m \le k$$

from M into the set M′ of partitions of n+k into exactly k parts. This mapping is bijective, since

i) two distinct k-tuples in M are mapped onto two distinct k-tuples in M′;

ii) every k-tuple in M′ is the image of a k-tuple of M. This is because, if $(p_1, p_2,\ldots,p_k)$ is a partition of n+k with k parts, then it is the image of $(p_1 - 1, p_2 - 1,\ldots, p_k - 1)$ under the mapping above.

Therefore, $|M| = P_n^1 + \ldots + P_n^k = |M'| = P_{n+k}^k$, and the theorem is proved.

Note that $P_n^k = 0$ **if n < k**, since there is no partition of n with k parts if n < k. Also, $\mathbf{P_n^n = P_n^1 = 1}$.

The formula in Theorem 1 is an identity which allows us to find P_n^r from values of P_m^k, where m < n, k ≤ r. This is why it is also called a **recurrence relation** for P_n^k.

Theorem 1 is every useful. For instance, to verify your count in E2, you can use it because $P_{10}^5 = P_5^1 + P_5^2 + \ldots + P_5^5 = 7$.

From Theorem 1, the P_n^ks may be calculated recursively as shown in Table 1.

Table 1 : P_n^k for 1 ≤ n, k ≤ 6

n \ k	1	2	3	4	5	6
1	1	0	0	0	0	0
2	1	1	0	0	0	0
3	1	1	1	0	0	0
4	1	2	1	1	0	0
5	1	2	2	1	1	0
6	1	3	3	2	1	1

In this table, the second entry in the row corresponding to n = 4 is P_4^2. By Theorem 1, $P_4^2 = P_2^1 + P_2^2$, which is the sum of the entries in the row corresponding to n = 2.

Similarly, P_6^3 is the sum of the entries in the row corresponding to n = 3.

Now, here is an exercise for you.

E3) Use Table 1 to find the values of $P_7^k, 1 \le k \le 6$.

The partition of a number n into k parts also tells us how n objects can be distributed among k boxes. We will now consider all possibilities of such distributions.

4.3 DISTRIBUTIONS

By a distribution we mean a way of placing several objects into a number of containers. For example, consider the distribution of 6 balls among 3 boxes. We may have all 6 balls of different shapes, sizes and colours, i.e., they are all distinguishable. Or, all the balls could be exactly the same, i.e., they are all indistinguishable.

Similarly, all 3 boxes may look different, or all 3 could be exactly the same. So, we see that there are 4 possibilities here.

In fact, we have the following possibilities for any set of n objects and m boxes.

Case 1: The objects are distinguishable, and so are the boxes;

Case 2: The objects are distinguishable and the boxes are indistinguishable;

Case 3: The objects are indistinguishable and the boxes are distinguishable;

Case 4: The objects are indistinguishable, and so are the boxes.

You may be surprised to know that in each of the cases the number of such distributions is different. In fact, the distribution problem is to count all possible distributions in any of these situations, or in a combination of these cases.

A general guideline for modelling a 'distribution problem' is that a distribution of distinct objects corresponds to an arrangement, and a distribution of identical objects corresponds to a selection. Let us consider examples of each of the four cases given above.

(a) There are twenty students and four colleges. In how many ways can the students be accommodated in the four colleges?

In this example the students, as well as the colleges, are clearly distinguishable. This comes under Case (1).

(b) Suppose we want to group 100 students into 10 groups of 10 each for the purpose of a medical examination. In how many ways can this be done?

Here the groups are indistinguishable, though the students in them are distinguishable. Hence, this falls under Case (2).

(c) An employer wants to distribute 100 one-rupee notes among 6 employees. What is the number of ways of doing this?

Though the one-rupee notes can be distinguished by their distinct numbers, we don't consider them to be distinguishable as far as their use is concerned. The employees, of course, are distinguishable. Hence, this is an example of Case (3).

(d) There are 1000 one-rupee notes. In how many ways can they be bundled into 20 bundles?

As before, the rupee notes are treated as indistinguishable. Clearly, the bundles are, by themselves, not distinguishable. Only the quantity in each may vary. Hence, this falls under Case (4).

Let us consider each case in some detail now.

4.3.1 Distinguishable Objects into Distinguishable Containers

Let us consider the example (A) above. Since any number of students can be put in a college, there are $20 \times 20 \times 20 \times 20$ possibilities, by the multiplication principle.

More generally, suppose we are distributing n objects into m containers, both being distinguishable. Then the total number of such distributions is n^m.

Let us look at an example.

Example 1: Show that the number of words of length n on an alphabet of m letters is m^n.

Solution: The m letters of the alphabet can be used any number of times in a word of n letters. The word can be considered as n ordered boxes, each holding a letter from the alphabet. The boxes become distinguishable because they are 'ordered'. The letters of the alphabet are clearly distinguishable. So, the number of ways of doing this is m^n.

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Several people are confused while solving the problem above. They tend to take the m letters as the containers instead! Let's consider another example.

Example 2: Suppose we have a set S with n objects. An m-sample from this set S is an ordered arrangement of m letters taken from S, with replacement at every draw, in m draws. Find the number of m-samples from an n-element set.

Solution: Every m-sample is a word of length m from the 'alphabet' S containing n letters. Hence, the required number is n^m.

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Now here are some exercises for you to solve.

E4) Find the number of three-letter words that can be formed using the letters of the English alphabet. How many of them end in x? How many of them have a vowel in the middle position?

E5) How many five-digit numbers are even? How many five-digit numbers are composed of only odd digits?

E6) There are 4 women and 5 men. A committee of three, a president, a vice-president, and a secretary, has to be formed from them. In how many ways can this be done if

i) the vice-president should be a woman?

ii) exactly one out of the vice-president and the secretary should be a woman?

iii) there is at least one woman in the committee?

Now suppose, we want to find the number of distributions of n distinguishable objects into m distinguishable containers, with **the extra condition** that no container should

contain more than one object. It is clear that this requires $m \geq n$. Then we can get all these arrangements by first choosing n containers to contain exactly one object, and then permuting the n objects among the chosen containers. This can be done in $C(m, n). \; n! = P(m, n)$ ways.
So, we have proved the following result.

Theorem 2: The number of ways of distributing n distinguishable objects into m distinguishable containers such that no container contains more than one object is P(m, n).

For example, the cardinality of the set of 5-digit numbers with all digits being distinct odd numbers is P (5,5). This is because the possible digits are 1, 3, 5, 7, 9.

Why don't you try an exercise now?

E7) Find the number of m-letter words with distinct letters, all taken from an alphabet with n letters, where $n \geq m$. Is this different from the number of injective mappings from an m-element set into an n-element set, where $n \geq m$? Give reasons for your answer.

Let us now consider the second type of distribution.

4.3.2 Distinguishable Objects into Indistinguishable Containers

Here we shall find the number of ways of distributing n distinguishable objects into m indistinguishable containers. For this, we first find the number when exactly k of the containers are occupied. This brings us to Stirling numbers of the second kind, named after James Stirling (1692-1770).

Suppose $n \geq m$. The number of distributions of n distinguishable objects into m indistinguishable containers **such that no container is empty** is represented by $\mathbf{S_n^m}$. This number is called the Stirling number of the second kind. As you can see, this is also the number of partitions of a set of n objects into m classes.

Definition: For natural numbers n and m, the **Stirling number of the second kind**, S_n^m, is the number of partitions of an n-element set into exactly m parts.

Note that:

i) $S_n^m = 0$ if $n < m$, for, if the number of containers exceeds the number of objects, then it is impossible to have all the containers non-empty.

ii) $S_n^n = 1$, since there is only one way of putting n distinguishable objects in n indistinguishable boxes so that no box is empty.

iii) $S_n^1 = 1$.

Now, we shall use the inclusion exclusion principle to find the value of S_n^m.

Theorem 3: $S_n^m = \dfrac{1}{m!}\sum\limits_{k=0}^{m}(-1)^k C(m, m-k)(m-k)^n$, $n \geq m$.

Proof: If the m classes are distinguishable, the number of partitions is the same as the number of functions from an n-element set onto an m-element set. As the classes are distinguishable here, we have to divide this number by m!. The result follows from Theorem 8, Unit 3.

For example, to obtain the Stirling number, S_5^3, we know that the number of functions from a 5-element set onto a three-element set is 150. So, by Theorem 3, $S_5^3 = 150/3!$ = 25.

Remark: You may be wondering how we have jumped straightaway to the Stirling numbers of the second kind. What about the first kind? We won't be using them in any way here. However, for a the sake of completeness, we define **Stirling numbers of the first kind, s(n, k),** as follows.

For a positive integer n, and $0 \le k \le n$, s(n, k) is the coefficient of x^k in the expansion of the multinomial $x(x-1)(x-2)\ldots(x-n+1)$.

Getting back to S_n^m, you may feel that the formula in Theorem 3 is a little cumbersome. Sometimes, the following recurrence relation for S_n^m may be more useful.

Theorem 4: If $1 < m \le n$, then $S_{n+1}^m = S_n^{m-1} + mS_n^m$.

Proof: Let us take n+1 objects, mark one of them, and consider the distribution of these n+1 objects into m indistinguishable containers. Then we have 2 situations.

Case (1) (The marked object is placed in one container without any other objects.): In this case, the remaining n objects can be placed in (m – 1) containers in S_n^{m-1} ways.

Case (2) (The marked object is placed with at least one more object in a container.): In this case, we can first distribute the n unmarked objects into m containers, and then put the marked objects m to one of these m containers. So, the number of such partitions is mS_n^m.

Therefore, by the addition principle, we get $S_{n+1}^m = S_n^{m-1} + mS_n^m$.

There is a generalisation of Theorem 4 that is of independent interest, which we now state.

Theorem 5: $S_{n+1}^m = \sum\limits_{k=0}^{n} C(n,k).S_k^{m-1}$

Proof: Let us mark one object in a set of (n+1) objects. Suppose the marked object is present in a box with (n – k + 1) elements, where $m - 1 \le k \le n$. Then we can choose n – k more objects to go with the marked object in C(n, n – k) ways. The remaining k objects can be distributed into $(m-1)^n$ boxes in S_k^{m-1} ways. So the number of ways of distributing the n – k objects is C(n, n – k) S_k^{m-1}. The result now follows from the addition principle by allowing k to vary from 0 to n.

Let us see some examples of the use of these recurrences.

Example 3: Calculate S_3^2 and S_4^2.

Solution: Using Theorem 4, we get $S_3^2 = S_2^1 + 2 \times S_2^2 = 1 + 2 \times 1 = 3$, and $S_4^2 = S_3^1 + 2S_3^2 = 1 + 2 \times 3 = 7$.

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Now let us find what we had started with in this sub-section.

Theorem 6: The number of ways of distributing n distinguishable objects into m indistinguishable containers is $S_n^1 + S_n^2 + \ldots + S_n^m$, where $n \geq m$. (Note that here we do not insist that no container is empty.)

Proof: When we distribute n distinguishable objects into m indistinguishable containers there are m cases. Case (k) is that exactly k containers are non-empty. Here k varies from 1 to m. The number of distributions in Case (k) is S_n^k. The result now follows from the addition principle.

Let us consider an example.

Example 4: In how many ways can 20 students be grouped into 3 groups?

Solution: Theorem 6 says that this can be done in $S_{20}^1 + S_{20}^2 + S_{20}^3$ ways.

Now, using Theorem 3, we get this number to be

$$1 + \frac{1}{2}\sum_{k=0}^{2}(-1)^k C(2,2-k)(2-k)^{20} + \frac{1}{6}\sum_{k=0}^{3}(-1)^k C(3,3-k)(3-k)^{20}$$

= 581,130,734.

* * *

Try some exercises now.

E8) Find the number of surjective functions from an n-element set onto an m-element set.

E9) Find the number of ways of placing n people in n – 1 rooms, no room being empty.

Let us now consider the third possibility for distributing objects into containers.

4.3.3 Indistinguishable Objects into Distinguishable Containers

Suppose there are n indistinguishable objects and m distinguishable containers. As the objects are indistinguishable, the distributions depend only on the number of objects in each container. As the containers are distinguishable, they can be assumed to be arranged in a line. Hence, the number of distributions is the number of ways of writing the number n as the sum $x_1+x_2+\ldots+x_m$, where the x_i's are non-negative integers.

We have covered this situation in Theorem 5 of Unit 2. Over there we have shown that **the number of distributions of n indistinguishable objects into m distinguishable containers is C (m+n–1, n)**. In particular, the number of non-negative integral solutions of the equation $x_1+x_2+\ldots+x_m = n$ is C (m+n–1, n).

Incidentally, we note that the number of distributions of n indistinguishable objects into m distinguishable containers **with at most one object per container** is C (m, n).

Let us consider an example.

Example 5: How many distinct solutions are there of x + y + z + w = 10

i) in non-negative integers?

ii) in positive integers?

Solution:

i) From the result quoted above, the answer is C(4 + 10 – 1, 10) = 286.

ii) We want x, y, z, w to be positive. Hence, we can write them respectively as X+1, Y+1, Z+1, W+1, where X, Y, Z, W are non-negative. Hence we want the number of non-negative solutions of the equation X+1+Y+1+Z+1+W+1=10, i.e., X+Y+Z+W= 6. The answer, now, is C (4 + 6 – 1, 6) = 84.
Try some exercises now.

E10) Show that the number of positive solutions of the equation $x_1+x_2+\ldots+x_n = m$ is C(m – 1, m – n).

E11) In how many ways can an employer distribute 100 one-rupee notes among 6 employees so that each gets at least one note?

Let us now consider the fourth case.

4.3.4 Indistinguishable Objects into Indistinguishable Containers

Suppose there are n indistinguishable objects and m indistinguishable containers. Any distribution is determined purely by an **unordered** m-tuple of non-negative integers with sum n. This is equivalent to the number of increasing sequences of length m of non-negative integers with sum n. But this is precisely the number of partitions of the integer n with at most m parts, viz., $P_n^1 + P_n^2 + \ldots + P_n^m = P_{n+m}^m$, from Theorem 1 of this unit.

Let us consider an example of this case.

Example 6: In how many ways can 20 identical books be placed in 4 identical boxes?

Solution: The answer is $P_{20}^1 + P_{20}^2 + P_{20}^3 + P_{20}^4 = P_{24}^4$

Why don't you try some exercises now?

E12) In how many ways can 1000 one-rupee notes be bundled into a maximum of 20 bundles?

E13) A car manufacture has 5 service centres in a city. 10 identical cars were served in these centres for a particular mechanical defect. In how many ways could the cars have been distributed at the various centres?

With this we have come to the end of this unit. Let us take a quick look at what we have studied in this unit.

4.4 SUMMARY

1. A partition of $n \in \mathbf{N}$ into k parts is $x_1+x_2+\ldots+x_k = n$, where $x_1 \le x_2 \le \ldots \le x_k$. P_n is the set of all partitions of n, and P_n^k is the set of all partitions into exactly k parts.

2. The proof and applications of the recurrence relation,
$P_n^1 + P_n^2 + \ldots + P_n^k = P_{n+k}^k, P_n^1 = P_n^n = 1, 1 \le k \le n.$

3. The number of ways of distributing n objects into m containers is :

i) n^m, if the objects and containers are distinguishable.

ii) $\sum_{i=1}^{m} S_n^i$, if the objects are distinguishable but the containers are not.

(Here S_j^i is a Stirling number of the second kind).

iii) C(m+n–1, n) if the objects are not distinguishable but the containers are distinguishable.

iv) P_{n+m}^m, if neither the objects nor the containers are distinguishable.

Further, in (i) above, if there is an extra requirement that each container contain at most one object, then the number of distributions is P(m, n). Again, in (iii) above, with the same extra requirement, the number of distributions is C(m, n).

4.5 SOLUTIONS /ANSWERS

E1) In the table below we give all possible partitions of 7.

Table 2

Number of parts	Partitions
1	7
2	1+6, 2+5, 3+4
3	1+1+5, 1+2+4, 1+3+3, 2+2+3
4	1+1+1+4, 1+1+2+3, 1+2+2+2
5	1+1+1+1+3, 1+1+1+2+2
6	1+1+1+1+1+2
7	1+1+1+1+1+1+1

From the table, we see that $P_7^4 = 3$, $P_7^5 = 2$.

E2) The required number is $P_{10}^5 = 7$.

E3) $P_7^1 = 1 = P_7^7$.

$P_7^2 = P_5^1 + P_5^2 = 1 + 2 = 3$, from Table 1.

$P_7^3 = P_4^1 + P_4^2 + P_4^3 = 1 + 2 + 1 = 4$, from Table 1.

Similarly, $P_7^4 = P_3^1 + P_3^2 + P_3^3 + P_3^4 = 3$, $P_7^5 = P_2^1 + P_2^2 = 2$ and $P_7^6 = P_1^1 = 1$.

E4) The 26 letters are distinguishable objects. We have to fill then in three distinguishable containers, viz., the first, second, and third positions of a three-lettered word. The solution is 26^3.

If the last letter is to be x, the number is only $26^2 \times 1$.

If the middle letter is a vowel, then by the multiplication principle, the answer is $26 \times 5 \times 26$.

E5) The total number of even numbers is $9 \times 10 \times 10 \times 10 \times 5 = 45{,}000$, since the last digit can only by 0, 2, 4, 6 or 8.

The number of 5-digit numbers composed of only odd digits (i.e., 1, 3, 5, 7, 9) is clearly $5^5 = 3125$.

E6) i) We can choose a woman for vice-president in 4 ways. To fill the remaining 2 positions we can select 2 from the remaining 8 persons in $8 \times 7 = 56$ ways. Hence, the required number is $4 \times 56 = 224$.

ii) If the vice-president is a woman (chosen in 4 ways), others can be selected in $5 \times 4 = 20$ ways. Similarly, if the woman is a secretary, the others can be chosen in 20 ways. Hence, by the addition and multiplication principles, the answer is $20 \times 4 + 20 \times 4 = 160$.

iii) Without any restriction, three can be selected in $9 \times 8 \times 7 = 504$ ways. If no woman is to be selected, then it can be done in $5 \times 4 \times 3 = 60$ ways. What we need is the complement of this. Thus, the required answer is $504 - 60 = 444$.

E7) If the alphabet has n letters, the m-letter words with distinct letters can be formed in $n(n-1)(n-2)\ldots(n-m+1) = P(n, m)$ ways.

Now, in an injective mapping, images of distinct elements should be distinct (see Unit 1). There are n possible images for the first element of the m-set, $n-1$ possible images for the second, and so on. Hence, the number of such mappings is also P(n, m).

E8) Suppose $N = \{1, 2, \ldots, n\}$ and $M = \{1, 2, \ldots, m\}$. If f is an onto function from N to M, then the inverse images, $f^{-1}(1), f^{-1}(2), \ldots, f^{-1}(k)$ constitute a partition of N into m classes. The number of ways in which this can be done is S_n^m, where the order of partition is immaterial. But, in functions, the order cannot be ignored. So, the distribution can be done in $m!.\,S_n^m$ ways.

E9) This is S_n^{n-1}. This can be done by putting one person each in $n-2$ rooms and 2 persons in 1 room. This can be done in C(n, 2) ways. So $S_n^{n-1} = C(n,2)$.

E10) If a positive solution is $x_1, x_2, \ldots, x_n$, then it can be written as $X_1+1, X_2+1, \ldots, X_n+1$, where the X_i's are non-negative. Thus, the required number is the number of non-negative solutions of $X_1+X_2+\ldots+X_n+n = m$, which is $C(n+m-n-1, m-n) = C(m-1, m-n)$.

E11) This is the number of positive solutions of $x_1+\ldots+x_6 = 100$.
So, the required number is $C(100-1, 100-6) = C(99, 94) = 71{,}523{,}144$.

E12) $P_{1000}^{1} + P_{1000}^{20} + P_{1020}^{20}$.

Had the requirement been that there be exactly 20 bundles, then the number would have been P_{1000}^{20}.

E13) $P_{10}^{1} + P_{10}^{2} + P_{10}^{3} + P_{10}^{4} + P_{10}^{5} = P_{15}^{5}$.